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Effects of chemical reaction on free convective flow of a polar fluid through a porous medium in the presence of internal heat generation

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Abstract

This paper is focused on the study of combined effects of free convective heat and mass transfer on the steady two-dimensional, laminar, polar fluid flow through a porous medium in the presence of internal heat generation and chemical reaction of the first order. The highly non-linear coupled differential equations governing the boundary layer flow, heat and mass transfer are solved by using two-term perturbation method with Eckert number E as perturbation parameter. The parameters that arise in the perturbation analysis are Eckert number E (viscous dissipation), Prandtl number Pr (thermal diffusivity), Schmidt number Sc (mass diffusivity), Grashof number Gr (free convection), solutal Grashof number Gm, chemical reaction parameter Δ (rate constant), internal heat generation parameter Q, material parameters α and β (characterizes the polarity of the fluid), C_f (skin friction coefficient), Nusselt number Nu (wall heat transfer coefficient) and Sherwood number Sh (wall mass transfer coefficient). Analytical expressions are computed numerically. Numerical results for the velocity, angular velocity, temperature and concentration profiles as well as for the skin friction coefficient, wall heat transfer and mass transfer rate are obtained and reported graphically for various conditions to show interesting aspects of the solution. Further, the velocity distribution of polar fluids is compared with the corresponding flow problems for a viscous (Newtonian) fluid and found that the polar fluid velocity is decreasing.

Keywords: Chemical reaction; Free convection; Polar fluid; Porous medium; Internal heat generation; Couple stress

1. Introduction

Many transport processes occurring both in nature and in industries involve fluid flows with the combined heat and mass transfer. Such flows are driven by the multiple buoyancy effects arising from the density variations caused by the variations in temperature as well as species concentrations. Convective flows in porous media have been extensively examined during the last several decades due to many practical applications which can be modeled or approximated as transport phenomena in porous media. References of comprehensive literature surveys regarding the subject of porous media can be had in most recent books by Ingham and Pop [1], Nield and Bejan [2], Vafai [3], Pop and Ingham [4] and Ingham et al. [5]. Coupled heat and mass transfer problems in presence of chemical reaction are of importance in many processes and have, therefore, received considerable amount of attention in recent times. In processes such as drying, distribution of temperature and moisture over agricultural fields and groves of fruit trees, damage of crops due to freezing, evaporation at the surface of a water body, energy transfer in a wet cooling tower and flow in a desert cooler, heat and mass transfer occur simultaneously. Possible applications of this type of flow can be found in many industries. For instance, in the power industry, among the methods of generating electric power is one in which electrical energy is extracted directly from a moving conducting fluid. Chemical reactions can be modeled as either homogeneous or heterogeneous processes. This depends on whether they occur at an interface or as a single phase volume reaction. A homogeneous reaction is one that occurs uniformly throughout a given phase. The species generation in a homogeneous reaction is the same as internal source of heat generation. On the other hand, a heterogeneous reaction takes place in a restricted area or within the boundary of a phase.

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Nomenclature

C'	species concentration
C'_w	surface concentration
C'_∞	species concentration far from the surface
C_a, C_d	are coefficients of couple stress viscosities
C_f	skin friction co-efficient
D	the effective diffusion co-efficient
E	Eckert number
C_p	specific heat at constant pressure
Gm	solutal Grashof number
Gr	Grashof number
g	acceleration due to gravity
I	a constant of dimension equal to that of the moment of inertia of unit mass
K'	permeability of the porous medium
K	dimensionless permeability
k_1	first order chemical reaction rate
Nu	Nusselt number
Pr	Prandtl number
Q_0	heat generation co-efficient
Q	heat generation parameter
Sc	Schmidt number
Sh	Sherwood number
T'	temperature in the boundary layer
T'_∞	temperature of the fluid far away from the plate
T'_w	temperature at the wall
U'_∞	free-stream velocity
u', v'	components of velocities along and perpendicular to the plate, respectively
u	dimensionless velocity
u_0	zeroth order velocity
u_1	first order velocity
v_0	suction velocity
X', Y'	co-ordinate system
y	dimensionless co-ordinate normal to the plate

Greek symbols

α, β	material parameters characterizing the polarity of the fluid
β_c	volumetric co-efficient of thermal expansion
β_t	volumetric co-efficient of expansion with concentration
Δ	chemical reaction parameter
γ	spin gradient velocity
λ	thermal conductivity of the fluid
ν	kinematic viscosity of the fluid
ν_r	rotational kinematic viscosity of the fluid
ρ	density of the fluid
ρ_∞	density of the fluid far away from the surface
Φ	dimensionless concentration
Θ	dimensionless temperature
Θ_0	zeroth order temperature
Θ_1	first order temperature
ω'	angular velocity component
ω	dimensionless angular velocity
ω_0	zeroth order angular velocity
ω_1	first order angular velocity

It can therefore be treated as a boundary condition similar to the constant heat flux condition in heat transfer. The study of heat and mass transfer with chemical reaction is of great practical importance to engineers and scientists because of its almost universal occurrence in many branches of science and engineering. Das et al. [6] have studied the effects of mass transfer on the flow past impulsively started infinite vertical plate with constant heat flux and chemical reaction. Diffusion of a chemically reactive species from a stretching sheet is studied by Andersson et al. [7]. Anjalidevi and Kandasamy [8,9] have analyzed the effects of chemical reaction, heat and mass transfer on laminar flow without or with MHD along a semi infinite horizontal plate. Muthucumaraswamy and Ganeshan [10–12] have studied the impulsive motion of a vertical plate with heat flux/mass flux/suction and diffusion of chemically reactive species. The flow and mass transfer on a stretching sheet with a magnetic field and chemically reactive species are examined by Takhar et al. [13]. Muthucumaraswamy [14] has analyzed the effects of a chemical reaction on a moving isothermal vertical surface with suction. Prasad et al. [15] have studied the effects of diffusion of chemically reactive species of a non-Newtonian fluid immersed in a porous medium over a stretching sheet. Ghaly and Seddeek [16] have discussed the Chebyshev finite difference method for the effects of chemical reaction, heat and mass transfer on laminar flow along a semi infinite horizontal plate with temperature dependent viscosity. Seddeek [17] has studied the finite element method for the effects of chemical reaction, variable viscosity, thermophoresis and heat generation/absorption on a boundary layer hydromagnetic flow with heat and mass transfer over a heat surface. Effects of chemical reaction, heat and mass transfer along a wedge with heat source and concentration in the presence of suction or injection are examined by Kandasamy et al. [18]. Raptis and Perdikis [19] have discussed the viscous flow over a non-linearly stretching sheet in the presence of chemical reaction and magnetic field.

Quite a number of physical phenomena involve free convection driven by heat generation. The study of heat generation in moving fluids is important in view of several physical problems such as those dealing with chemical reactions and those concerned with dissociating fluids. Possible heat generation effects may change the temperature distribution and, therefore, the particle deposition rate. This may occur in such applications related to nuclear reactor cores, fire and combustion modeling, electronic chips and semi conductor wafers. In fact, the literature is replete with examples dealing with heat transfer in laminar flow of viscous fluids. Vajravelu and Hadjinicolaou [20] have studied heat transfer characteristics in a laminar boundary layer flow of a viscous fluid over a linearly stretching continuous surface with viscous dissipation and internal heat generation. In this analysis they have considered the volumetric rate of heat generation, q''' [W/m^3] as

$$q''' = \begin{cases} Q_0(T' - T'_\infty) & \text{for } T \geqslant T_\infty \\ 0 & \text{for } T < T_\infty \end{cases}$$

where q''' is internal volumetric heat generation and Q_0 is the heat generation coefficient. Effect of heat generation or absorption on free convective flow with heat and mass transfer in geometries with and without porous media has been studied by many scientists and technologists [21–29].

Recently, interest in problems of non-Newtonian fluids has grown considerably because of more and more applications in chemical process industries, food preservation techniques, petroleum production and in power engineering. Many industrial fluids are non-Newtonian in nature and their characteristics are considered rheological. More particularly, slurries (china clay and coal in water, sewage sludge, etc.), multiphase mixtures (oil–water emulsions, gas–liquid dispersions such as froths and foams, butter etc.), are non-Newtonian fluids. Further more examples displaying a variety of non-Newtonian characteristics include pharmaceutical formulations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial fluid saliva etc.), and foodstuffs (jams, jellies, soaps, marmalades etc.). The non-Newtonian boundary layer flows through a porous medium with heat and mass transfer are seen in such wide applications as fluid film lubrication, analysis of polymer in chemical engineering etc. Fossil fuels may saturate underground beds and display non-Newtonian behavior. Several models have been proposed to explain the non-Newtonian behavior of fluids. The fluids which sustain couple stresses, called polar fluids, model those fluids with micro-structures which are mechanically significant when the characteristic dimension of the problem is of the same order of magnitude as the size of the micro-structure. Extensive reviews of the theory can be found in the review article by Cowin [30]. Since the micro-structure size is the same as the average pore size, it is relevant to study the flow of polar fluids through a porous medium. The examples of fluids which can be modeled as polar fluids are slurries, polymer solutions, mud, crude oil, body fluids, lubricants with polymer additives, etc. The effects of couple stresses on the flow through a porous medium have been investigated by Raptis [31] and Patil and Hiremath [32]. Hiremath and Patil [33] have analyzed the free convection effects on the oscillatory flow of couple stress fluids through a porous medium. In this analysis, the effect of couple stresses in the Darcy resistance was not considered. Sharma and Gupta [34] examined the effects of thermal convection in micropolar fluids in porous medium. Effect of rotation on thermal convection in micropolar fluids through a porous medium have studied by Sharma and Kumar [35]. Raptis and Takhar [36] have examined steady flow of a polar fluid through a porous medium by using the generalized Forchheimer's model. Sharma and Thakur [37] have analysed the effects of MHD on couple stress fluid heated from below in porous medium. Sharma and Sharma [38] have discussed effects of the thermosolutal convection of micropolar fluids with MHD through a porous medium. Kim [39] has examined the unsteady convection flow effects of micropolar fluids through a porous medium in which, free-stream velocity consisting of a mean velocity and temperature with super imposed exponentially small variation with time is considered. Kim [40] has analysed the unsteady MHD convection flow of polar fluids past a vertical moving porous plate in a porous medium. Ibrahim et al. [41] have examined the effects of unsteady MHD micropolar fluid flow over a vertical porous plate through a porous medium in the presence of thermal and mass diffusion with a constant heat source. The effect of heat and mass transfer on MHD micropolar flow over a vertical moving porous plate in a porous medium has studied by Kim [42]. Hassanein et al. [43] have examined the effects of natural convection flow of micropolar fluid from a permeable uniform heat flux surface in porous medium. Sharma and Sharma [44] have discussed the couple stress fluid permeated with suspended particles heated and soluted from below in porous medium. Ogulu [45] has studied the effect of oscillating temperature flow of a polar fluid past a vertical porous plate in the presence of couple stresses and radiation. Rehman and Sattar [46] have analyzed the effects of magnetohydrodynamic convective flow of a micropolar fluid past a continuously moving porous plate in the presence of heat generation/absorption. The effect of rotation on a layer of micropolar ferromagnetic fluid heated from below saturating a porous medium are investigated by Sunil et al. [47].

In the present paper, the Authors propose to study the effects of chemical reaction and internal heat generation on the free convective flow with heat and mass transfer of a polar fluid through porous medium in the presence of couple stresses. Examples of present physical flow situation are: (i) the heat removal of nuclear fuel debris buried in the deep sea-bed and (ii) heat recovery from geothermal systems. The flow configuration is modeled as hot vertical plate bounding the porous region filled with water containing soluble and insoluble chemical materials. Such a fluid is modeled as polar fluids. The flow is due to buoyancy forces generated by the temperature gradient. The analysis of the results obtained reveals that the flow field is influenced considerably by the presence of chemical reaction, internal heat source, viscous and Darcy's dissipation.

2. Mathematical analysis

We consider steady, laminar, two-dimensional free convective flow of a viscous incompressible polar fluid with mass transfer through a porous medium occupying a semi-infinite region of the space bounded by an infinite vertical porous plate. The X' is taken along the vertical plate and Y' is normal to it. The velocity, the angular velocity, the temperature and the species concentration fields are $(u', v', 0)$, $(0, 0, \omega')$, T' and C' respectively. The surface is maintained at a constant temperature T'_w different from the porous medium temperature T'_∞ sufficiently away from the surface and allows a constant suction. A heat source is placed within the flow to allow possible heat generation effects. The concentration of diffusing species is assumed to be very small in comparison with other chemical species which are present, the concentration of species far from the surface C'_∞, is infinitely small [48] and hence the

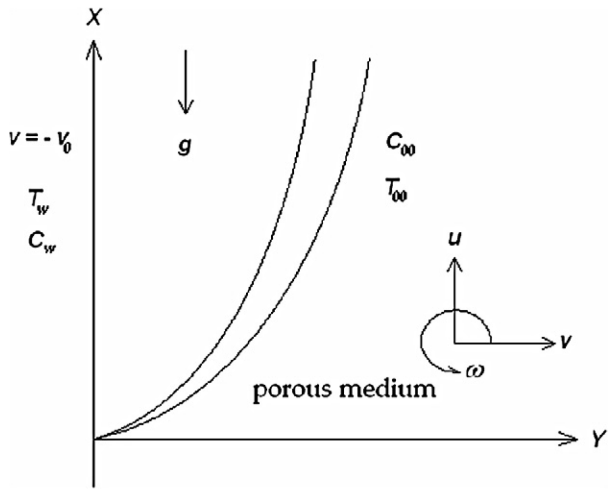

Fig. 1. Physical model and co-ordinate system.

Soret and Dufour effects are neglected. However, the effects of the viscous dissipation and Darcy dissipation (ignoring the contribution due to couple stresses as a first approximation) are accounted in the energy balance equation. The chemical reactions are taking place in the flow and all thermo physical properties are assumed to be constant. The flow is due to buoyancy effects arising from density variations caused by differences in the temperature as well as species concentration. The governing equations for this physical situation are based on the usual balance laws of mass, linear momentum, angular momentum, energy and mass diffusion modified to account for the physical effects mentioned above. These equations are given by

$$\frac{\partial v'}{\partial y'} = 0 \tag{1}$$

$$v'\frac{\partial u'}{\partial y'} = g\beta_t(T' - T'_\infty) + g\beta_c(C' - C'_\infty) + (\nu + \nu_r)\frac{\partial^2 u'}{\partial y'^2} + 2\nu_r\frac{\partial \omega'}{\partial y'} - \frac{\nu + \nu_r}{K'}u' \tag{2}$$

$$v'\frac{\partial \omega'}{\partial y'} = \frac{\gamma}{I}\frac{\partial^2 \omega'}{\partial y'^2} \tag{3}$$

$$v'\frac{\partial T'}{\partial y'} = \frac{\lambda}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} + \frac{\nu}{C_p}\left(\frac{\partial u'}{\partial y'}\right)^2 + \frac{\nu}{K'C_p}u'^2 + \frac{Q_0}{\rho C_p}(T' - T'_\infty) \tag{4}$$

$$v'\frac{\partial C'}{\partial y'} = D\frac{\partial^2 C'}{\partial y'^2} - k_1 C' \tag{5}$$

after employing the "volume averaging" process [49], and $\gamma = (C_a + C_d)/I$. Following D'ep [50], the appropriate boundary conditions are:

$$y' = 0:\quad u' = 0,\ \frac{\partial \omega'}{\partial y'} = -\frac{\partial^2 u'}{\partial y'^2},\ T' = T'_w,\ C' = C'_w$$
$$y' \to \infty:\quad u' \to 0,\ \omega' \to 0,\ T' \to C'_w,\ C' \to C'_\infty \tag{6}$$

The boundary conditions (6) are derived from the assumption that the couple stresses are dominant during the rotation of the particles.

The integration of the continuity equation (1) yields

$$v' = -v_0 \tag{7}$$

where v_0 the constant suction velocity at the wall and the negative sign indicates that the suction velocity is directed towards the plate.

Introducing the following non-dimensional quantities:

$$y = y'v_0/\nu,\qquad u = u'/v_0,\qquad \Theta = (T' - T'_\infty)/(T'_w - T'_\infty)$$
$$\Phi = (C' - C'_\infty)/(C'_w - C'_\infty),\qquad \omega = \omega'\nu/v_0,\qquad \alpha = \nu_r/\nu$$
$$\beta = I\nu/\gamma,\qquad Pr = \rho\nu C_p/\lambda,\qquad E = v_0^2/C_p(T'_w - T'_\infty)$$
$$Gr = \nu g\beta_t(T'_w - T'_\infty)/v_0^3,\qquad Gm = \nu g\beta_c(C'_w - C'_\infty)/v_0^3$$
$$K = K'v_0^2/\nu^2,\qquad Q = Q_0\nu/\rho C_p v_0^2,\qquad Sc = \nu/D$$
$$\Delta = k_1\nu/v_0^2 \tag{8}$$

into Eqs. (2)–(5) and in view of Eq. (7), we obtain the non-dimensional equations:

$$(1+\alpha)u'' + u' - \frac{1+\alpha}{K}u = -\{Gr\,\Theta + Gm\,\Phi + 2\alpha\omega'\} \tag{9}$$

$$\omega'' + \beta\omega' = 0 \tag{10}$$

$$\Theta'' + Pr\,\Theta' - Pr\,Q\Theta = -Pr\,E\left\{u'^2 + \frac{u^2}{K}\right\} \tag{11}$$

$$\Phi'' + Sc\,\Phi' - Sc\,\Delta\Phi = 0 \tag{12}$$

and the boundary conditions (6) reduce to the non-dimensional form:

$$y = 0:\quad u = 0,\ \omega' = -u'',\ \Theta = 1,\ \Phi = 1$$
$$y \to \infty:\quad u \to 0,\ \omega \to 0,\ \Theta \to 0,\ \Phi \to 0 \tag{13}$$

where prime denotes the differentiation with respect to y. The mathematical statement of the problem is now complete and now move on to obtain the solution of Eqs. (9)–(12) subject to boundary conditions (13).

3. Solution of the problem

The solution of Eq. (12) subject to the corresponding boundary conditions (13) is

$$\Phi(y) = \mathrm{Exp}(R_1 y) \tag{14}$$

where $R_1 = \frac{-Sc - \sqrt{Sc^2 + 4Sc\,\Delta}}{2}$.

The problem posed in Eqs. (9)–(11) subject to the boundary conditions presented in Eq. (13) are highly non-linear, coupled equations and generally will involve a step by step numerical integration of the explicit finite difference scheme. However, analytical solutions are possible. Since viscous dissipation parameter E is very small in most of the practical problems and therefore, we can advance an asymptotic expansion with E as perturbation parameter for the velocity, angular velocity and temperature as follows:

$$u(y)=u_0(y)+Eu_1(y)+\mathrm{O}(E^2) \tag{15}$$

$$\omega(y)=\omega_0(y)+E\omega_1(y)+\mathrm{O}(E^2) \tag{16}$$

$$\Theta(y)=\Theta_0(y)+E\Theta_1(y)+\mathrm{O}(E^2) \tag{17}$$

where the zeroth order terms correspond to the case in which the viscous and Darcy's dissipation is neglected ($E=0$). The substitution of Eqs. (15)–(17) into Eqs. (9)–(11) and the conditions (13), we get the following system of equations.

Zeroth-order:

$$(1+\alpha)u_0''+u_0'-\frac{1+\alpha}{K}u_0=-\{Gr\,\Theta_0+Gm\,\Phi+2\alpha\omega_0'\} \tag{18}$$

$$\omega_0''+\beta\omega_0'=0 \tag{19}$$

$$\Theta_0''+Pr\,\Theta_0'-Pr\,Q\Theta_0=0 \tag{20}$$

subject to the reduced boundary conditions

$$y=0:\quad u_0=0,\ \omega_0'=-u_0'',\ \Theta_0=1$$
$$y\to\infty:\quad u_0\to 0,\ \omega_0\to 0,\ \Theta_0\to 0 \tag{21}$$

First-order:

$$(1+\alpha)u_1''+u_1'-\frac{1+\alpha}{K}u_1=-\{Gr\,\Theta_1+2\alpha\omega_1'\} \tag{22}$$

$$\omega_1''+\beta\omega_1'=0 \tag{23}$$

$$\Theta_1''+Pr\,\Theta_1'-Pr\,Q\Theta_1=-Pr\left\{u_0'^2+\frac{u^2}{K}\right\} \tag{24}$$

subject to the reduced boundary conditions

$$y=0:\quad u_1=0,\ \omega_1'=-u_1'',\ \Theta_1=0$$
$$y\to\infty:\quad u_1\to 0,\ \omega_1\to 0,\ \Theta_1\to 0 \tag{25}$$

By solving Eqs. (18)–(20) under the conditions (21) and Eqs. (22)–(24) under the conditions (25) and then substituting the obtained solutions in Eqs. (15)–(17), we obtain

$$\begin{aligned}u(y)&=C_2\,\mathrm{Exp}(R_5y)+A_1\,\mathrm{Exp}(R_3y)+A_2\,\mathrm{Exp}(R_1y)\\&\quad+A_3\,\mathrm{Exp}(-\beta y)\\&\quad+E\left\{\begin{array}{l}C_4\,\mathrm{Exp}(R_5y)+A_{14}\,\mathrm{Exp}(R_3y)\\+A_{15}\,\mathrm{Exp}(2R_5y)+A_{16}\,\mathrm{Exp}(2R_3y)\\+A_{17}\,\mathrm{Exp}(2R_1y)+A_{18}\,\mathrm{Exp}(-2\beta y)\\+A_{19}\,\mathrm{Exp}((R_3+R_5)y)\\+A_{20}\,\mathrm{Exp}((R_1+R_5)y)\\+A_{21}\,\mathrm{Exp}((R_5-\beta)y)\\+A_{22}\,\mathrm{Exp}((R_1+R_3)y)\\+A_{23}\,\mathrm{Exp}((R_3-\beta)y)\\+A_{24}\,\mathrm{Exp}((R_1-\beta)y)\\+A_{25}C_3\,\mathrm{Exp}(-\beta y)\end{array}\right\}\\&\quad+\mathrm{O}(E^2)\end{aligned} \tag{26}$$

$$\omega(y)=\{C_1+EC_3\}\,\mathrm{Exp}(-\beta y)+\mathrm{O}(E^2) \tag{27}$$

$$\begin{aligned}\Theta(y)&=\mathrm{Exp}(R_3y)+E\left\{\begin{array}{l}D_1\,\mathrm{Exp}(R_3y)+A_4\,\mathrm{Exp}(2R_5y)\\+A_5\,\mathrm{Exp}(2R_3y)\\+A_6\,\mathrm{Exp}(2R_1y)\\+A_7\,\mathrm{Exp}(-2\beta y)\\+A_8\,\mathrm{Exp}((R_3+R_5)y)\\+A_9\,\mathrm{Exp}((R_1+R_5)y)\\+A_{10}\,\mathrm{Exp}((R_5-\beta)y)\\+A_{11}\,\mathrm{Exp}((R_1+R_3)y)\\+A_{12}\,\mathrm{Exp}((R_3-\beta)y)\\+A_{13}\,\mathrm{Exp}((R_1-\beta)y)\end{array}\right\}\\&\quad+\mathrm{O}(E^2)\end{aligned} \tag{28}$$

where $R_{3,4}=\frac{-Pr\mp\sqrt{Pr^2+4Pr\,Q}}{2}$ and $R_{5,6}=\frac{-1\mp\sqrt{1+\frac{4}{K}(1+\alpha)^2}}{2}$.

From the engineering point of view, the most important characteristics of the flow are the skin friction co-efficient C_f, Nusselt Nu and Sherwood Sh numbers, which are given below:

$$\begin{aligned}C_f&=\left[\frac{\mathrm{d}u}{\mathrm{d}y}\right]_{y=0}=\left[\frac{\mathrm{d}u_0}{\mathrm{d}y}+E\frac{\mathrm{d}u_1}{\mathrm{d}y}\right]_{y=0}\\&=R_5C_2+R_3A_1+R_1A_2-\beta A_3\end{aligned} \tag{29}$$

$$+E\left\{\begin{array}{l}R_5C_4+R_3A_{14}+2R_5A_{15}+2R_3A_{16}\\+2R_1A_{17}-2\beta A_{18}+(R_3+R_5)A_{19}\\+(R_1+R_5)A_{20}+(R_5-\beta)A_{21}\\+(R_1+R_3)A_{22}+(R_3-\beta)A_{23}\\+(R_1-\beta)A_{24}-\beta C_3A_{25}\end{array}\right\} \tag{30}$$

$$Nu=-\left[\frac{\mathrm{d}\Theta}{\mathrm{d}y}\right]_{y=0}=-\left[\frac{\mathrm{d}\Theta_0}{\mathrm{d}y}\right]_{y=0}-E\left[\frac{\mathrm{d}\Theta_1}{\mathrm{d}y}\right]_{y=0} \tag{31}$$

$$=-R_3-E\left\{\begin{array}{l}D_1R_3+2R_5A_4+2R_3A_5+2R_1A_6\\-2\beta A_7+(R_3+R_5)A_8\\+(R_1+R_5)A_9+(R_5-\beta)A_{10}\\+(R_1+R_3)A_{11}+(R_3-\beta)A_{12}\\+(R_1-\beta)A_{13}\end{array}\right\} \tag{32}$$

and

$$Sh=-\left[\frac{\mathrm{d}\Phi}{\mathrm{d}y}\right]_{y=0}=-R_1 \tag{33}$$

The remaining mathematical expressions involved in Eqs. (26)–(28) are given in Appendix A.

4. Results and discussion

In order to study the behavior of velocity u, angular velocity ω, temperature Θ and concentration Φ fields, a comprehensive numerical computation is carried out for various values of the parameters that describe the flow characteristics and the results are reported in terms of graphs. This is done in order to illustrate the special features of the solutions. To be more realistic, the values of Schmidt number (Sc) are chosen for hydrogen ($Sc=0.22$), water vapor ($Sc=0.62$) and ammonia ($Sc=0.78$) at temperature 25 °C and one atmospheric pressure. The values of Prandtl number (Pr) is chosen to be $Pr=0.71, 2.0, 5.0, 7.0, 10.0$. The effect of buoyancy is significant for $Pr=0.71$ (air) due to the lower density. Grashof number (Gr) for heat transfer is chosen to be $Gr=0.0, 2.0, 4.0$, where $Gr=0.0$ corresponds to the no free convection currents and for other values the physical situation corresponds to

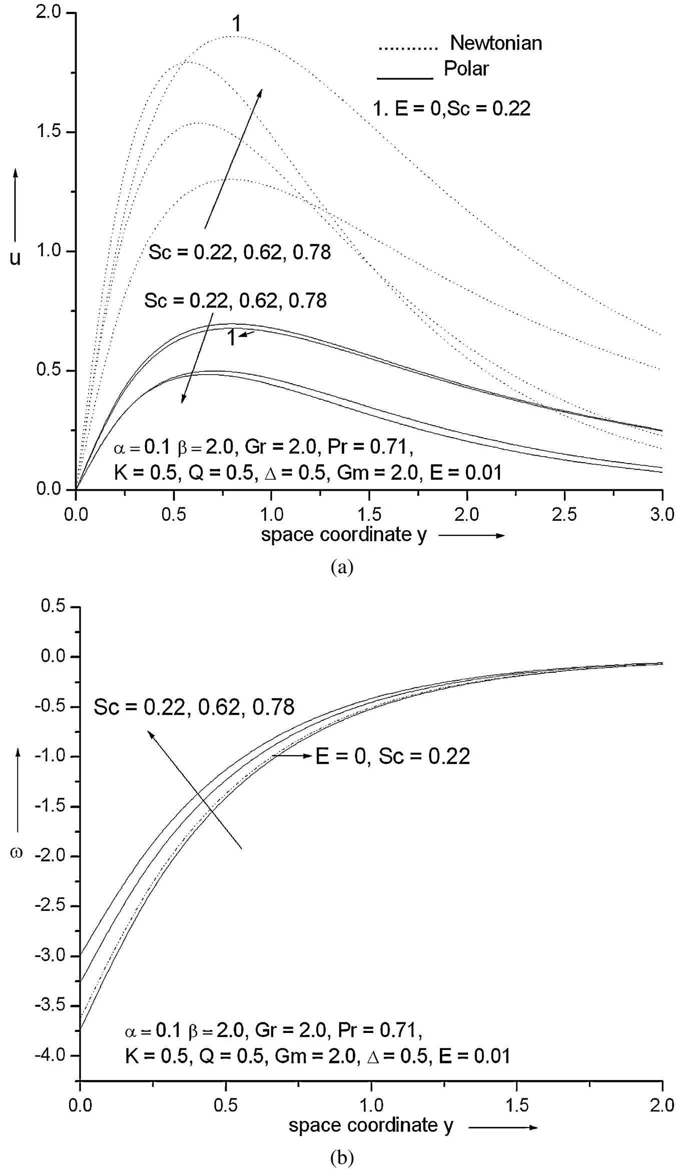

Fig. 2. Effects of Schmidt number *Sc* on velocity profiles.

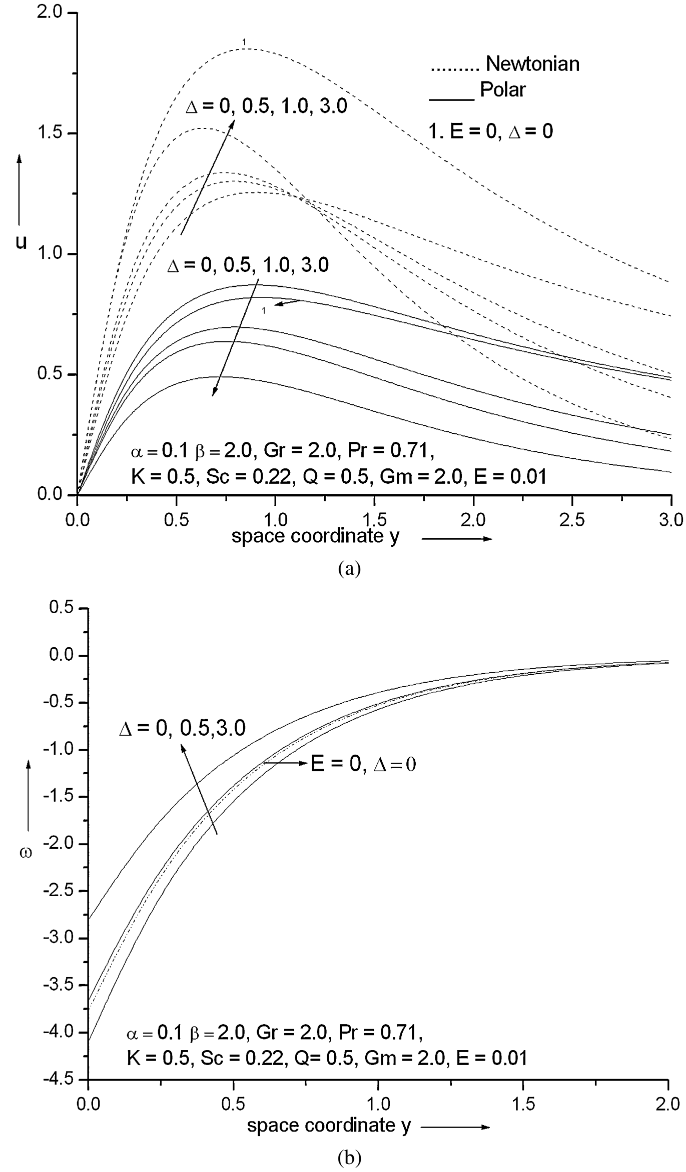

Fig. 3. Effects of chemical reaction parameter Δ on velocity profiles.

the case of cooling of the surface. The solutal Grashof number (Gm) for mass transfer $Gm = 0.0, 2.0, 4.0$, corresponds to cooling ($Gr > 0$) of the surface. The internal heat generation parameter Q is chosen to be $Q = 0.0, 0.5, 1.0, 2.0, 3.0$, where $Q = 0.0$ corresponds to the case of no heat source and chemical reaction parameter $\Delta = 0.0, 0.5, 1.0, 2.0, 3.0$, where $\Delta = 0.0$ corresponds to the case of no chemical reaction. Due to non-availability of experimental data for couple stress parameters (α and β), the suitable representative values are chosen in order to determine the polar effects on the flow characteristics.

Figs. 2 to 4 present typical profiles of the velocity and angular velocity for various values of Schmidt number (Sc), chemical reaction parameter Δ and the internal heat generation parameter Q, respectively. It is noted from Fig. 2 that an increase in the values of Schmidt number (Sc), leads to a fall in the velocity of the polar fluid whereas in the case of Newtonian (viscous) fluid, leads to a rise in the velocity. Further, it is observed that the velocity of a polar fluid is found to decrease in comparison with Newtonian ($\alpha = 0$ and $\beta = 0$) fluid. The opposite behavior is found in the case of angular velocity. The negative values of the angular velocity indicate that the micro rotation of sub structures in the polar fluid is clock-wise. This very behavior is also observed in the variation of chemical reaction parameter Δ from Fig. 3. The effects of internal heat generation parameter Q on the velocity and angular velocity is displayed in Fig. 4. Due to homogeneous chemical reaction and constant suction, it is clear that as the parameter Q increases, in both the cases of polar and Newtonian fluids, the velocity and angular velocity (in magnitude) leads to a fall. Further, it is noticed that the velocity of a Newtonian fluid is reduced remarkably as compared to the polar fluid. The influence of internal heat generation is more on the Newtonian fluid than that of a polar fluid. It is important to note that, in all the cases, the effects of viscous and Darcy's dissipation are ignored ($E = 0$) which leads to a fall in the velocity

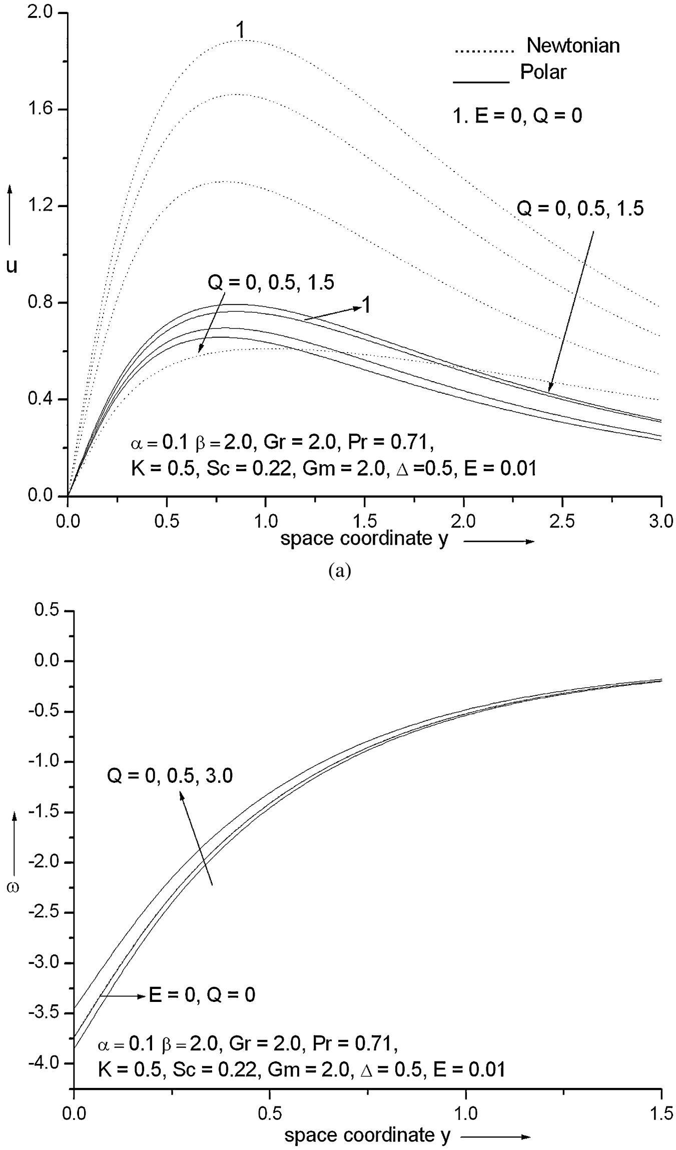

Fig. 4. Effects of heat generation parameter Q on velocity profiles.

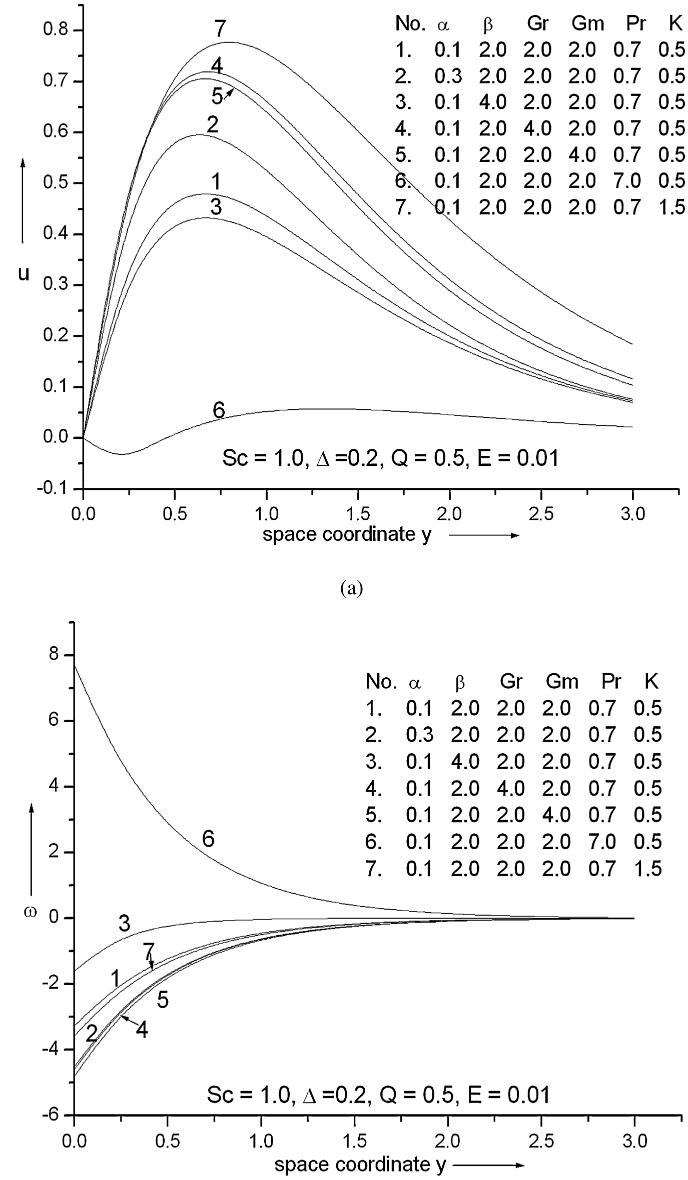

Fig. 5. Velocity profiles for various parameters.

of polar fluid while they enhance in the case of Newtonian fluid. To enhance the velocity of a polar fluid, viscous and Darcy's dissipation must be taken into account in the energy balance law. The effects of viscous and Darcy's dissipation may become very important in several flow configurations occurring in the engineering practice. In fact viscous and Darcy's dissipation affects strongly the heat transfer process whenever the operating fluid a low thermal conductivity and a high viscosity. The effect of velocity and angular velocity of the material (polar) parameters α and β, Grashof number Gr, solutal Grashof number Gm, permeability of the porous medium K and Prandtl number Pr are observed from the Fig. 5. The velocity and angular velocity fields are influenced by the combined effect of chemical reaction, internal heat generation, constant suction and viscous and Darcy's dissipation. We observe that velocity increases as α, Gr, Gm and K increase while it decreases as β and Pr increase. The opposite behavior is found in the case of angular velocity. From Eq. (26), we observe that the velocity boundary layer has multiple boundary layer structure with thicknesses of order R_5^{-1}, R_3^{-1}, R_1^{-1} and β^{-1}. There exist sub layers essentially due to the interaction among the velocity, the angular velocity, the temperature and concentration fields.

Figs. 6 to 8 illustrate the influence of heat generation, chemical reaction and Schmidt number on the dimensionless temperature Θ. Owing to the presence of homogeneous chemical reaction and constant suction, the thermal state of the fluid decreases causing lower thermal boundary layers as heat generation parameter Q increases. The effects of chemical reaction and Schmidt number Sc are less as compared to the heat source. The facts are seen from the Figs. 7 and 8. The reason is that the viscous and Darcy's dissipation terms are of second degree in velocity and velocity gradient. The Schmidt number Sc and chemical reaction parameter Δ effects on velocity field are not

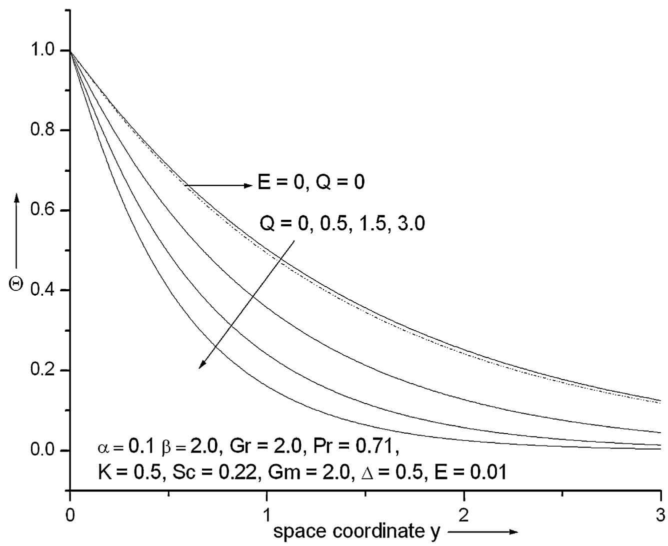

Fig. 6. Effects of heat generation parameter Q on temperature profiles.

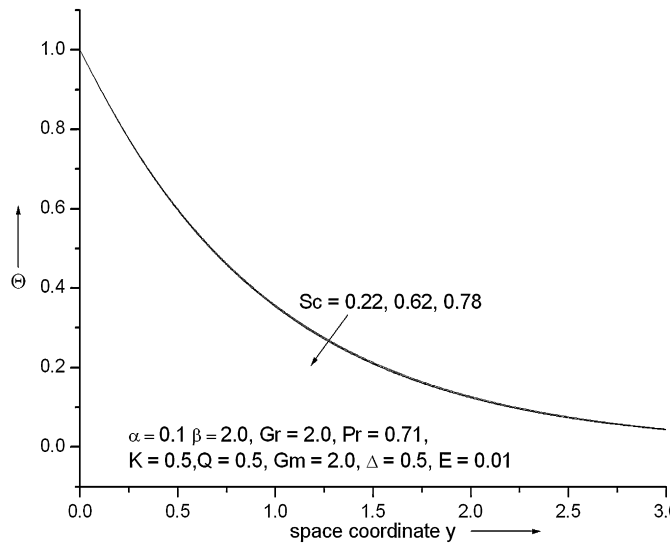

Fig. 8. Effects of Schmidt number Sc on temperature profiles.

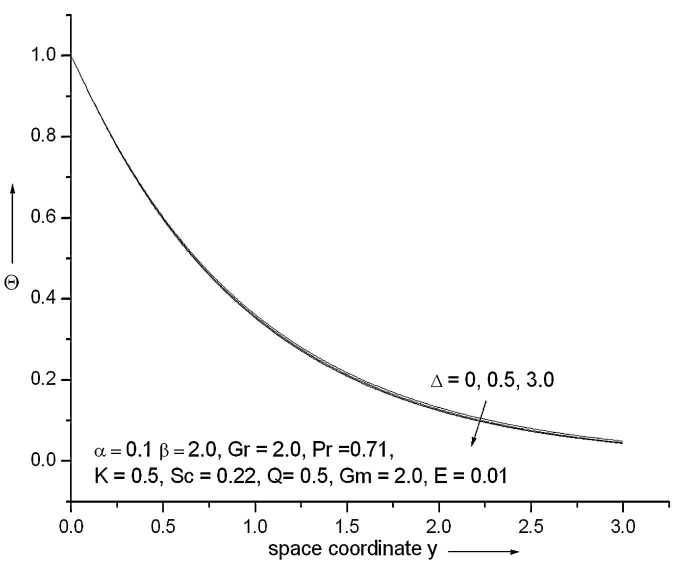

Fig. 7. Effects of chemical reaction parameter Δ on temperature profiles.

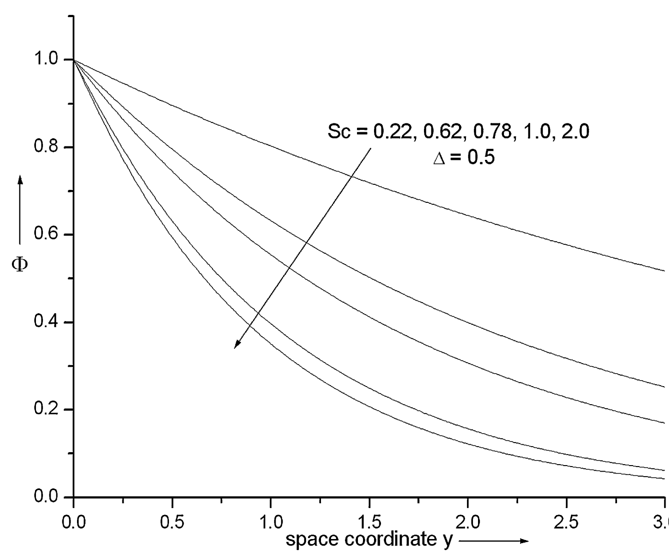

Fig. 9. Effects of Schmidt number Sc on concentration profiles.

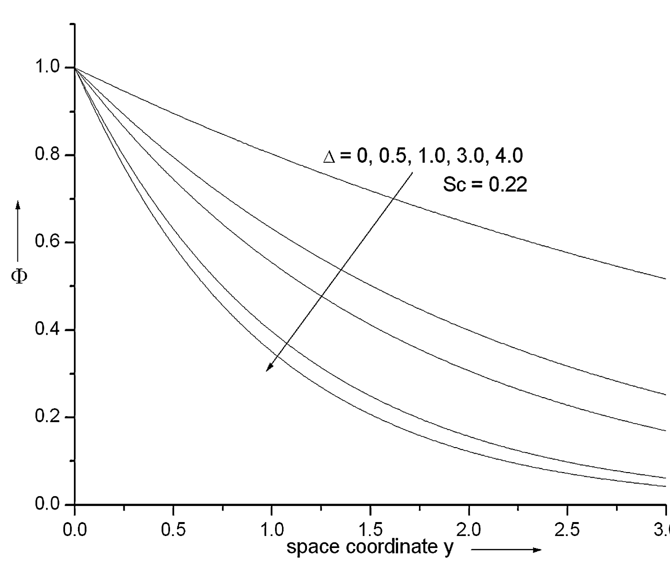

Fig. 10. Effects of chemical reaction parameter Δ on concentration profiles.

significant enough to affect the temperature field considerably. The relative influence of the polar (material) parameters α and β, permeability of the porous medium K, Grashof number Gr, solutal Grashof number Gm and Prandtl number Pr are also investigated on the fluid temperature but the results are not presented herein for brevity. It is observed from these results that the temperature distribution is slightly increased as α, Gr, Gm and K increase while it is decreased as β and Pr increase as a result of viscous and Darcy's dissipation effect which acts as a heat source and chemical reaction of first order. Eq. (28) indicates the multiple boundary layer structure of thermal boundary layer with dominant layer of order R_3^{-1}. Also there exist sub layers due to the interaction of temperature field with the velocity, the angular velocity and the concentration fields.

Figs. 9 and 10 show typical concentration profiles for various values of the Schmidt number Sc and the chemical reaction parameter Δ, respectively. It is clear from Figs. 9 and 10

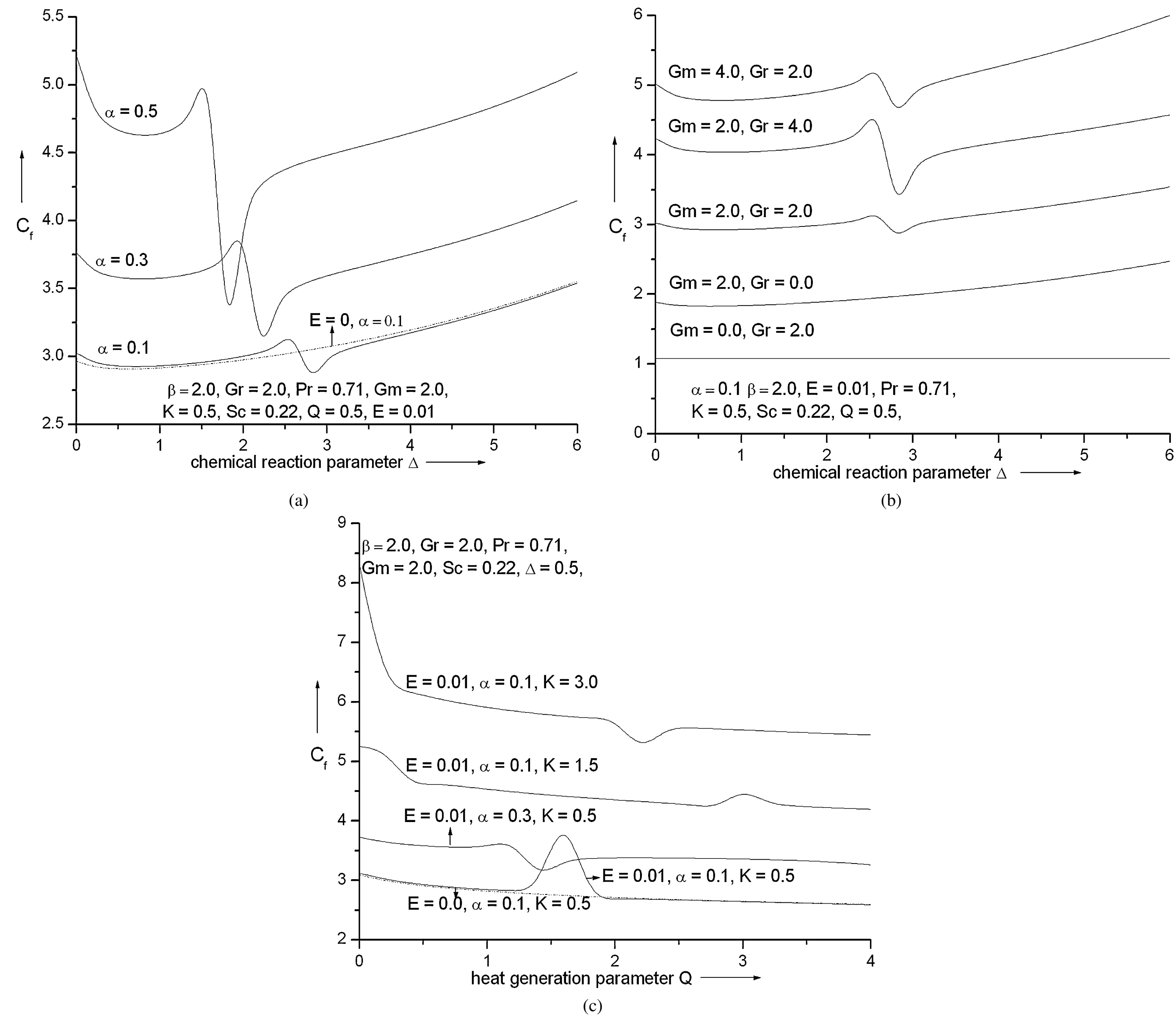

Fig. 11. (a) Effects of polar parameter α on skin friction against chemical reaction parameter Δ. (b) Effects of *Gr* and *Gm* on skin friction against chemical reaction parameter Δ. (c) Effects of polar parameter α and permeability K on skin friction against heat generation parameter Q.

that the concentration boundary layer thickness decreases as the Schmidt number *Sc* and chemical reaction parameter Δ increase; this is analogous to the effect of increasing the Prandtl number *Pr* on the thickness of a thermal boundary layer. Eq. (14) implies that the species diffusion boundary layer thickness of order R_1^{-1}.

The skin friction (C_f), the wall heat transfer (*Nu*) and the wall mass transfer (*Sh*) coefficients are plotted in Figs. 11 to 13, respectively. In Fig. 11, skin friction coefficient is plotted against chemical reaction parameter Δ and heat generation parameter Q. It is observed that the wall slope of the velocity increases against the chemical reaction parameter Δ, as the polar parameter α, Grashof number *Gr* and solutal Grashof number *Gm* increase (Fig. 11(a), (b)) while it decreases with K and increases with α as the heat generation parameter Q increases (Fig. 11(c)). The variation of skin friction coefficient C_f with chemical reaction parameter Δ shows anomalous behavior at a critical chemical reaction parameter. As material parameter α increases this anomalous behavior is more significant and is observed at lower critical chemical reaction parameter. For example $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$ the critical chemical reaction parameter $\Delta = 2.5, 2.0, 1.5$, respectively. This similar observation is found in figure (b) and (c) also. Fig. 12 shows that the wall slope of the temperature profile increases as either *Pr* or Q increases. This is consistent with Fig. 12. Finally, Fig. 13 display the influence of chemical reaction parameter Δ on the wall mass transfer in terms of Sherwood number *Sh* in the absence of viscous and Darcy's dissipations. It is observed that wall mass transfer increases as either chemical reaction parameter Δ or Schmidt number *Sc* is increased.

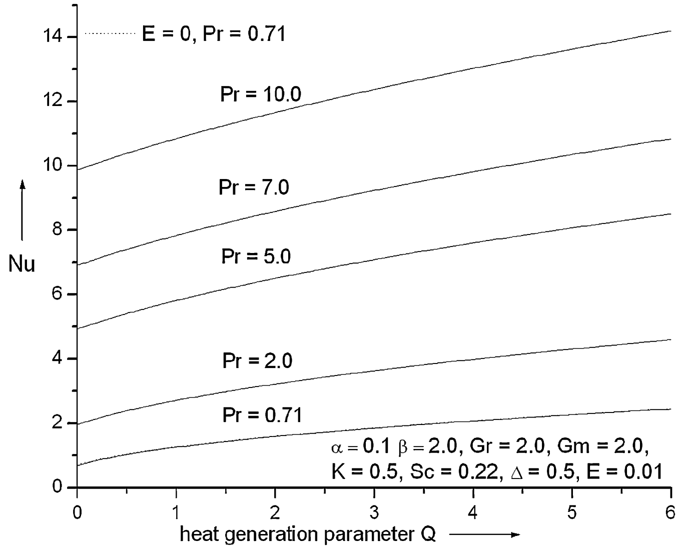

Fig. 12. Effects of Prandtl number Pr on wall heat transfer Nu against heat generation parameter Q.

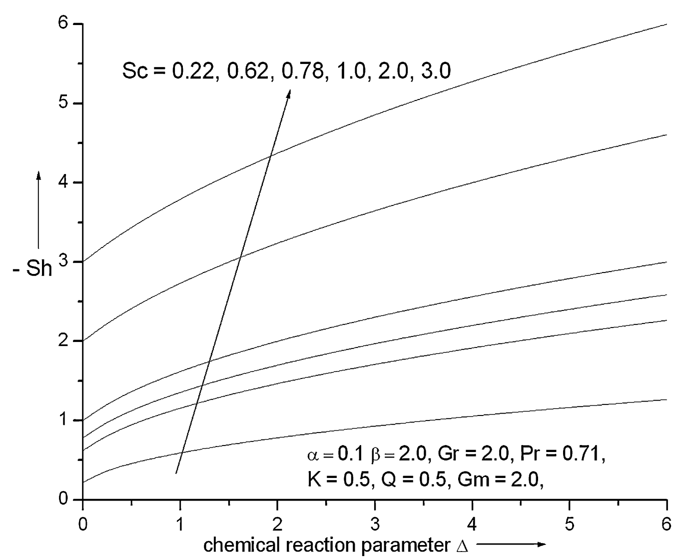

Fig. 13. Effects of Schmidt number Sc on wall mass transfer Sh against chemical reaction parameter Δ.

5. Conclusions

The work considered herein reflects the effects of chemical reaction, constant suction and internal heat generation on steady, laminar, viscous incompressible free convective flow of a polar fluid through a porous medium occupied by a semi infinite region of the space over a porous flat surface. A set of non-linear coupled differential equations governing the fluid velocity, angular velocity, temperature and species concentration is solved by two-term perturbation method. The parameters that arise in the perturbation analysis are Eckert number E (viscous dissipation), Prandtl number Pr (thermal diffusivity), Schmidt number Sc (mass diffusivity), Grashof number Gr (free convection), solutal Grashof number Gm, chemical reaction parameter Δ (rate constant), internal heat generation parameter Q, material parameters α and β (characterizes the polarity of the fluid), C_f skin friction coefficient, Nu Nusselt number Nu (wall heat transfer coefficient) and Sherwood number Sh (wall mass transfer coefficient). A comprehensive set of graphical results for the velocity, angular velocity, temperature and concentration is presented and discussed. It is found that the velocity of the polar fluid is considerably reduced as compared to the Newtonian fluid in the presence of combined effects of chemical reaction, constant suction, internal heat generation and viscous and Darcy's dissipation.

Appendix A

The mathematical expressions appeared in Eqs. (26)–(28) are as follows.

$$A_1 = \frac{-Gr}{(1+\alpha)(R_3-R_5)(R_3-R_6)}$$
$$A_2 = \frac{-Gm}{(1+\alpha)(R_1-R_5)(R_1-R_6)}$$
$$A_3 = \frac{2\alpha\beta}{(1+\alpha)(\beta+R_5)(\beta+R_6)}$$
$$A_4 = \frac{-Pr\,C_2^2(R_5^2+K^{-1})}{(2R_5-R_3)(2R_5-R_4)}$$
$$A_5 = \frac{-Pr\,A_1^2(R_3^2+K^{-1})}{R_3(2R_3-R_4)}$$
$$A_6 = \frac{-Pr\,A_2^2(R_1^2+K^{-1})}{(2R_1-R_3)(2R_1-R_4)}$$
$$A_7 = \frac{-Pr\,A_3^2C_1^2(\beta^2+K^{-1})}{(2\beta+R_3)(2\beta+R_4)}$$
$$A_8 = \frac{-2\,Pr\,A_1C_2(R_3R_5+K^{-1})}{R_5(R_3+R_5-R_4)}$$
$$A_9 = \frac{-2\,Pr\,A_2C_2(R_1R_5+K^{-1})}{(R_1+R_5-R_3)(R_1+R_5-R_4)}$$
$$A_{10} = \frac{-2\,Pr\,A_3C_1C_2(K^{-1}-R_5\beta)}{(R_5-\beta-R_3)(R_5-\beta-R_4)}$$
$$A_{11} = \frac{-2\,Pr\,A_1A_2(R_1R_3+K^{-1})}{R_1(R_1+R_3-R_4)}$$
$$A_{12} = \frac{2\,Pr\,A_1A_3C_1(K^{-1}-R_3\beta)}{\beta(R_3-\beta-R_4)}$$
$$A_{13} = \frac{-2\,Pr\,A_2A_3C_1(K^{-1}-R_1\beta)}{(R_1-\beta-R_3)(R_1-\beta-R_4)}$$
$$A_{14} = \frac{-Gr\,D_1}{(1+\alpha)(R_3-R_5)(R_3-R_6)}$$
$$A_{15} = \frac{-Gr\,A_4}{(1+\alpha)R_5(2R_5-R_6)}$$
$$A_{16} = \frac{-Gr\,A_5}{(1+\alpha)(2R_3-R_5)(2R_3-R_6)}$$
$$A_{17} = \frac{-Gr\,A_6}{(1+\alpha)(2R_1-R_5)(2R_1-R_6)}$$

$$A_{18} = \frac{-Gr\,A_7}{(1+\alpha)(2\beta + R_5)(2\beta + R_6)}$$
$$A_{19} = \frac{-Gr\,A_8}{(1+\alpha)R_3(R_3 + R_5 - R_6)}$$
$$A_{20} = \frac{-Gr\,A_9}{(1+\alpha)R_1(R_1 + R_5 - R_6)}$$
$$A_{21} = \frac{Gr\,A_{10}}{(1+\alpha)\beta(R_5 - \beta - R_6)}$$
$$A_{22} = \frac{-Gr\,A_{11}}{(1+\alpha)(R_1 + R_3 - R_5)(R_1 + R_3 - R_6)}$$
$$A_{23} = \frac{-Gr\,A_{12}}{(1+\alpha)(R_3 - \beta - R_5)(R_3 - \beta - R_6)}$$
$$A_{24} = \frac{-Gr\,A_{13}}{(1+\alpha)(R_1 - \beta - R_5)(R_1 - \beta - R_6)}$$
$$A_{25} = A_3 C_3$$
$$C_1 = \frac{A_1(R_5^2 - R_3^2) + A_2(R_5^2 - R_1^2)}{A_3(\beta^2 - R_5^2) - \beta}$$
$$C_2 = \frac{A_3(R_3^2 A_1 + R_1^2 A_2) + (A_1 + A_2)(\beta - \beta^2 A_3)}{A_3(\beta^2 - R_5^2) - \beta}$$
$$C_3 = \frac{H_2 - H_1 R_5^2}{A_{25}(\beta^2 - R_5^2) - \beta}$$
$$C_4 = \frac{H_1(\beta^2 A_{25} - \beta) - H_2 A_{25}}{A_{25}(\beta^2 - R_5^2) - \beta}$$
$$D_1 = -\sum_{i=4}^{13} A_i$$
$$H_1 = -\sum_{j=14}^{24} A_j$$
$$H_2 = -\left\{\begin{array}{l} R_3^2 A_{14} + 4R_5^2 A_{15} + 4R_3^2 A_{16} + 4R_1^2 A_{17} + 4\beta^2 A_{18} \\ + (R_3 + R_5)^2 A_{19} + (R_1 + R_5)^2 A_{20} \\ + (R_5 - \beta)^2 A_{21} + (R_1 + R_3)^2 A_{22} \\ + (R_3 - \beta)^2 A_{23} + (R_1 - \beta)^2 A_{24} \end{array}\right\}$$

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